1.5 Solution Sets of Linear Systems

Definition. *A system of linear equations is said to be* homogeneous *if it can be written in the form* $A\mathbf{x} = \mathbf{0}$*, where* A *is an* $m \times n$ *matrix and* $\mathbf{0}$ *is the zero vector in* $\mathbb{R}^m$*.*

Such a system *always* has at least one solution, namely $\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^n$. This zero solution is usually called the *trivial* solution. For a homogeneous equation, the important question is whether there exists a nontrivial solution, that is, a nonzero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$. By the Existence and Uniqueness Theorem, we have the following fact:

Fact. *The homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *has a nontrivial solution if and only if the equation has at least one free variable.*

Example 1. *Determine if the following system has a nontrivial solution.*

$$\begin{aligned} x_1 - 3x_2 + 7x_3 &= 0 \\ -2x_1 + x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 + 9x_3 &= 0 \end{aligned}$$

Suppose $\mathbf{u}, \mathbf{v}$ are two vectors that span a plane. Suppose s, t are arbitrary scalars. The equation of the form $\mathbf{x} = t\mathbf{v}$ is called a parametric vector equation of a line through the origin. The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ is called a parametric vector equation of a plane.

Example 2. *Write the solution set of the following system in parametric vector form.*

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

Example 3. *Describe the solutions of the following system in parametric vector form and give a geometric description of the solution set.*

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

The equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}, t \in \mathbb{R}$ describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form. Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector $\mathbf{p}$ to the solutions of $A\mathbf{x} = \mathbf{0}$. The vector $\mathbf{p}$ is just one particular solution of $A\mathbf{x} = \mathbf{b}$. Geometrically, given $\mathbf{v}$ and $\mathbf{p}$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to move $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. Thus $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ is the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$. The relation between the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ generalizes to any consistent equation $A\mathbf{x} = \mathbf{b}$, although the solution set will be larger than a line when there are several free variables.

Theorem 6. *Suppose the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent for some given* $\mathbf{b}$*, and let* $\mathbf{p}$ *be a solution. Then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$*, where* $\mathbf{v}_h$ *is any solution of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$*.*

Theorem 6 says that if $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ for the translation.

To write a solution set of a consistent system in parametric vector form, we may follow the following steps:

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Example 4. *Mark each statement True or False. Justify each answer.*

1. *A homogeneous equation is always consistent.*
2. *The equation* $A\mathbf{x} = \mathbf{0}$ *has the trivial solution if and only if the equation has at least one free variable.*
3. *The homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *gives an explicit description of its solution set.*
4. *The equation* $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ *describes a line through* $\mathbf{v}$ *parallel to* $\mathbf{p}$.
5. *The solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, *where* $\mathbf{v}_h$ *is any solution of the equation* $A\mathbf{x} = \mathbf{0}$.