

## EQUATIONS FOR SEQUENCES

## 5.2.1 through 5.2.3

In these lessons, students learn multiple representations for sequences: as a string of numbers, as a table, as a graph, and as an equation. Read more about writing equations for sequences in the Math Notes box in Lesson 7.2.3.

In addition to the ways to write explicit equations for sequences, as explained in the Math Notes box in Lesson 7.2.3, equations for sequences can also be written recursively. An explicit formula tells exactly how to find any specific term in the sequence. A recursive formula names the first term (or any other term) and how to get from one term to the next. For an explanation of recursive sequences, see the Math Notes box in Lesson 5.3.2.

**Example 1**

This is the same scenario as in Example 1 of the previous section, *Introduction to Sequences*.

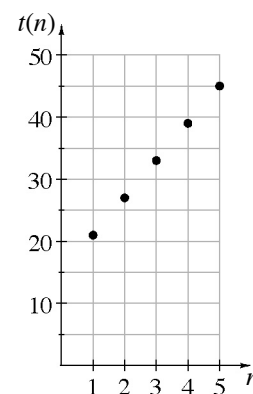
Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Write a sequence for the number of houses built, then write an equation for the sequence. Fully describe a graph of this sequence.

The sequence is 21, 27, 33, 39, .... Note that sequences usually begin with the first term, where the number of months is  $n = 1$ .

The common difference is  $m = 6$ , and the zeroth term is  $b = 15$ . The equation can be written  $t(n) = mn + b = 6n + 15$ . Note that for a sequence,  $t(n) =$  is used instead of  $y =$ .  $t(n) =$  indicates the equation is for a discrete sequence, as opposed to a continuous function. Students compared sequences to functions in Lesson 5.3.3.

The equation could also have been written as  $a_n = 6n + 15$ .

The graph of the sequence is shown at right. There are no  $x$ - or  $y$ -intercepts. There is no point at  $(0, 15)$  because sequences are usually written starting with the *first* term where  $n = 1$ . The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the  $y$ -values of the points that follow the rule  $t(n) = 6n + 15$  when  $n \geq 1$ . There are no asymptotes. The graph is linear and is shown at right. *This graph is discrete* (separate points). (Note: The related function,  $y = 6x + 15$ , would have the domain of all real numbers (including fractions and negatives) and the graph would be a continuous connected line.)



## Example 2

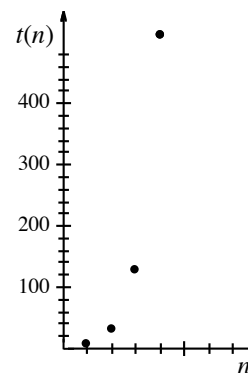
This is the same scenario as in Example 2 of the previous section, *Introduction to Sequences*.

When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. Write a sequence for the number of people who knew about Rosa mishap in ten-minute intervals, then write an equation for the sequence. Fully describe a graph of this sequence.

The multiplier is  $b = 4$ , and the zeroth term is  $a = 2$ . The equation can be written  $t(n) = ab^n = 2 \cdot 4^n$ . The equation could also have been written as  $a_n = 2 \cdot 4^n$ . (Later, in Chapter 7, students will also learn “first term” notation for sequences,  $a_n = 8 \cdot 4^{(n-1)}$ .)

The sequence is: 8, 32, 128, 512, ... . Note that the sequence is written starting with  $n = 1$ .

The graph of the sequence is to the right. There are no  $x$ - or  $y$ -intercepts. There is no point at  $(0, 2)$  because sequences are usually written starting with the *first* term where  $n = 1$ . The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the  $y$ -values of the points that follow the rule  $t(n) = 2(4)^n$  when  $n \geq 1$ . The graph is exponential and is shown at right. There is no symmetry. This graph is discrete (separate points). (Note: The related function,  $y = 2 \cdot 4^n$  would have the domain of all real numbers (including fractions and negatives) and the graph would be a connected curve.)



**Example 3**

Consider the two sequences:

$$\begin{array}{l} \text{A: } -8, -5, -2, 1, \dots \\ \text{B: } 256, 128, 64, \dots \end{array}$$

- For each sequence, is it arithmetic, geometric, or neither? How can you tell? Explain completely.
- What are the zeroth term and the generator for each sequence?
- For each sequence, write an equation representing the sequence.
- Is 378 a term of sequence A? Justify your answer.
- Is  $\frac{1}{4}$  a term of sequence B? Justify your answer.

To determine the type of sequence for A and B above, we have to look at the growth of each sequence.

$$\begin{array}{ccccccc} \text{A:} & -8, & -5, & -2, & 1, & \dots & \\ & \backslash & / & \backslash & / & \backslash & / \\ & & +3 & & +3 & & +3 \end{array}$$

Sequence A is made (generated) by adding three to each term to get the next term. When each term has a **common difference** (in this case, “+3”) the sequence is **arithmetic**.

Sequence B, however, is different. The terms do not have a common difference.

$$\begin{array}{ccccccc} \text{B:} & 256, & 128, & 64, & \dots & & \\ & \backslash & / & \backslash & / & & \\ & & -128 & & -64 & & \end{array}$$

Instead, these terms have a **common ratio** (multiplier). A sequence with a common ratio is a **geometric sequence**.

$$\begin{array}{ccccccc} \text{B:} & 256, & 128, & 64, & \dots & & \\ & \backslash & / & \backslash & / & & \\ & & \cdot \frac{1}{2} & & \cdot \frac{1}{2} & & \end{array}$$

The first term for sequence A is  $-8$ , and has a generator or common difference of  $+3$ . Therefore the zeroth term is  $-11$  (because  $-11 + 3 = -8$ ). An arithmetic sequence has an equation of the form  $t(n) = mn + b$  (or  $a_n = mn + a_0$ ) where  $m$  is the common difference, and  $b$  is the initial value. For sequence A, the equation is  $t(n) = 3n - 11$ , for  $n = 1, 2, 3, \dots$

*Example continues on next page →*

Example continued from previous page.

For sequence B, the first term is 256 with a generator or common ratio of  $\frac{1}{2}$ . Therefore the zeroth term is 512, because  $512 \cdot \frac{1}{2} = 256$ . The general equation for a geometric sequence is  $t(n) = ab^n$  where  $a$  is the zeroth term, and  $b$  is the common ratio (multiplier). For sequence B, the equation is  $t(n) = 512\left(\frac{1}{2}\right)^n$  for  $n = 1, 2, 3, \dots$

To check if 378 is a term in sequence A, we could list the terms of the sequence out far enough to check, but that would be time consuming. Instead, we will check if there is an integer  $n$  that solves  $t(n) = 3n - 11 = 378$ .

$$3n - 11 = 378$$

$$3n = 389$$

$$n = \frac{389}{3} = 129\frac{2}{3}$$

When we solve,  $n$  is not a whole number, therefore 378 cannot be a term in the sequence.

Similarly, to check if  $\frac{1}{4}$  is a term in sequence B, we need to solve  $t(n) = 512\left(\frac{1}{2}\right)^n = \frac{1}{4}$ , and look for a whole number solution.

$$512\left(\frac{1}{2}\right)^n = \frac{1}{4}$$

$$\frac{1}{512} \cdot 512\left(\frac{1}{2}\right)^n = \frac{1}{512} \cdot \frac{1}{4}$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{2048}$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{2^{11}}$$

$$\frac{1}{2^n} = \frac{1}{2^{11}}$$

$$n = 11$$

Although solving an equation like this is probably new for most students, they can solve this problem by using guess-and-check. Also, by writing both sides as a power of 2, students can see the solution easily.

Since the equation has a whole number solution,  $\frac{1}{4}$  is a term of sequence B.

That is, when  $n = 11$ ,  $t(n) = \frac{1}{4}$ .

## EXAMPLES FOR ARITHMETIC SEQUENCES

List the first five terms of each arithmetic sequence.

**Example 4** (An explicit formula)

$$t(n) = 5n + 2$$

$$t(1) = 5(1) + 2 = 7$$

$$t(2) = 5(2) + 2 = 12$$

$$t(3) = 5(3) + 2 = 17$$

$$t(4) = 5(4) + 2 = 22$$

$$t(5) = 5(5) + 2 = 27$$

The sequence is: 7, 12, 17, 22, 27, ...

**Example 5** (A recursive formula)

$$t(1) = 3, \quad t(n+1) = t(n) - 5$$

$$t(1) = 3$$

$$t(2) = t(1) - 5 = 3 - 5 = -2$$

$$t(3) = t(2) - 5 = -2 - 5 = -7$$

$$t(4) = t(3) - 5 = -7 - 5 = -12$$

$$t(5) = t(4) - 5 = -12 - 5 = -17$$

The sequence is: 3, -2, -7, -12, -17, ...

**Example 6**

Find an explicit and a recursive formula for the sequence:  $-2, 1, 4, 7, \dots$

Explicit:  $m = 3$ ,  $b = -5$ , so the equation is:  $t(n) = mn + b = 3n - 5$

Recursive:  $t(1) = -2$ ,  $t(n+1) = t(n) + 3$

**EXAMPLES FOR GEOMETRIC SEQUENCES**

List the first five terms of each geometric sequence.

**Example 7** (An explicit formula)

$$t(n) = 3 \cdot 2^{n-1}$$

$$t(1) = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3$$

$$t(2) = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6$$

$$t(3) = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12$$

$$t(4) = 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24$$

$$t(5) = 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48$$

The sequence is:  $3, 6, 12, 24, 48, \dots$

**Example 8** (A recursive formula)

$$t(1) = 8, \quad t(n+1) = t(n) \cdot \frac{1}{2}$$

$$t(1) = 8$$

$$t(2) = t(1) \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$$

$$t(3) = t(2) \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$$

$$t(4) = t(3) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$$

$$t(5) = t(4) \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

The sequence is:  $8, 4, 2, 1, \frac{1}{2}, \dots$

**Example 9**

Find an explicit and a recursive formula for the sequence:  $81, 27, 9, 3, \dots$

Explicit:  $a_1 = 81$ ,  $b = \frac{1}{3}$  so the  $a_0$  (the zeroth term) is found by  $a_0 = 81 \div \frac{1}{3} = 243$  and the equation is:  $a_n = a_0 \cdot b^n = 243 \cdot \left(\frac{1}{3}\right)^n$ , or alternatively,  $t(n) = 243 \cdot \frac{1}{3}^n$

Recursive:  $t(1) = 81$ ,  $t(n+1) = t(n) \cdot \frac{1}{3}$

## Problems

Each of the functions listed below defines a sequence. List the first five terms of the sequence, and state whether the sequence is arithmetic, geometric, both, or neither.

1.  $t(n) = 5n + 2$       2.  $s_n = 3 - 8n$       3.  $u(n) = 9n - n^2$       4.  $t(n) = (-4)^n$
5.  $s(n) = \left(\frac{1}{4}\right)^n$       6.  $u(n) = n(n + 1)$       7.  $t(n) = 8$       8.  $s_n = \frac{3}{4}n + 1$

Identify each of the following sequences as arithmetic or geometric. Then write the equation that gives the terms of the sequence.

9. 48, 24, 12, 6, 3, ...      10. -4, 3, 10, 17, 24, ...      11. 43, 39, 35, 31, 27, ...
12.  $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \frac{27}{128}, \dots$       13. 5, -5, 5, -5, 5, ...      14. 10, 1, 0.1, 0.01, 0.001, ...

Graph the following two sequences on the same set of axes.

15.  $t(n) = -6n + 20$       16. 1, 4, 16, 64, ...
17. Do the two sequences of the last two problems have any terms in common? Explain how you know.
18. Every year since 1548, the average height of a human male has increased slightly. The new height is 100.05% of what it was the previous year. If the average male's height was 54 inches in 1548, what was the average height of a male in 2008?
19. Davis has \$5.40 in his bank account on his fourth birthday. If his parents add \$0.40 to his bank account every week, when will he have enough to buy the new Smokin' Derby race car set which retails for \$24.99?
20. Fully describe the graph of the sequence  $t(n) = -4n + 18$ .

## Arithmetic Sequences

List the first five terms of each arithmetic sequence.

21.  $t(n) = 5n - 2$

22.  $t(n) = -3n + 5$

23.  $t(n) = -15 + \frac{1}{2}n$

24.  $t(n) = 5 + 3(n - 1)$

25.  $t(1) = 5, t(n+1) = t(n) + 3$

26.  $t(1) = 5, t(n+1) = t(n) - 3$

27.  $t(1) = -3, t(n+1) = t(n) + 6$

28.  $t(1) = \frac{1}{3}, t(n+1) = t(n) + \frac{1}{2}$

Find the 30<sup>th</sup> term of each arithmetic sequence.

29.  $t(n) = 5n - 2$

30.  $t(n) = -15 + \frac{1}{2}n$

31.  $t(31) = 53, d = 5$

32.  $t(1) = 25, t(n+1) = t(n) - 3$

For each arithmetic sequence, find an explicit and a recursive formula.

33. 4, 8, 12, 16, 20, ...

34. -2, 5, 12, 19, 26, ...

35. 27, 15, 3, -9, -21, ...

36.  $3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{3}, \dots$

Sequences are graphed using points of the form: (term number, term value).

For example, the sequence 4, 9, 16, 25, 36, ... would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), .... Sequences are graphed as points and not connected.

37. Graph the sequences from problems 21 and 22 above and determine the slope of each line.

38. How does the slope of the line found in the previous problem relate to the sequence?

## Geometric Sequences

List the first five terms of each geometric sequence.

39.  $t(n) = 5 \cdot 2^n$

40.  $t(n) = -3 \cdot 3^n$

41.  $t(n) = 40 \left(\frac{1}{2}\right)^{n-1}$

42.  $t(n) = 6 \left(-\frac{1}{2}\right)^{n-1}$

43.  $t(1) = 5, t(n+1) = t(n) \cdot 3$

44.  $t(1) = 100, t(n+1) = t(n) \cdot \frac{1}{2}$

45.  $t(1) = -3, t(n+1) = t(n) \cdot (-2)$

46.  $t(1) = \frac{1}{3}, t(n+1) = t(n) \cdot \frac{1}{2}$

Find the 15<sup>th</sup> term of each geometric sequence.

47.  $t(14) = 232, r = 2$

48.  $t(16) = 32, r = 2$

49.  $t(14) = 9, r = \frac{2}{3}$

50.  $t(16) = 9, r = \frac{2}{3}$

Find an explicit and a recursive formula for each geometric sequence.

51. 2, 10, 50, 250, 1250, ...

52. 16, 4, 1,  $\frac{1}{4}$ ,  $\frac{1}{16}$ , ...

53. 5, 15, 45, 135, 405, ...

54. 3, -6, 12, -24, 48, ...

55. Graph the sequences from problems 39 and 52. Remember the note before problem 37 about graphing sequences.

56. How are the graphs of geometric sequences different from arithmetic sequences?

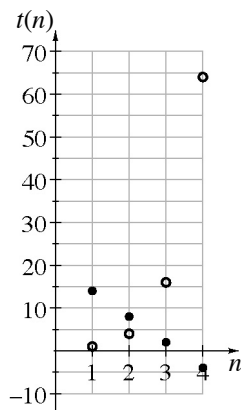


## Answers

1. 7, 12, 17, 22, 27, arithmetic, common difference is 5.
2.  $-5, -13, -21, -29, -37$ , arithmetic, common difference is  $-8$ .
3. 8, 14, 18, 20, 20, neither
4.  $-4, 16, -64, 256, -1024$ , geometric, common ratio is  $-4$ .
5.  $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$ , geometric, common ratio is  $\frac{1}{4}$ .
6. 2, 6, 12, 20, 30, neither
7. 8, 8, 8, 8, 8, both, common difference is 0, common ratio is 1.
8.  $\frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4, \frac{19}{4}$ , arithmetic, common difference is  $\frac{3}{4}$ .
9. geometric,  $t(n) = 96\left(\frac{1}{2}\right)^n$
10. arithmetic,  $t(n) = 7n - 11$
11. arithmetic,  $t(n) = -4n + 47$
12. geometric,  $t(n) = \left(\frac{8}{9}\right)\left(\frac{3}{4}\right)^n$
13. geometric,  $t(n) = -5(-1)^n$
14. geometric,  $t(n) = 100\left(\frac{1}{10}\right)^n$

15. See dots on graph.

16. See circles on graph.



17. No, they do not. The graph is discrete, or just points, which are the terms of each sequence. Since they do not share a common point, they do not have any terms in common.

18. In 2008,  $54(1.0005)^{460} \approx 67.96$  inches.
19.  $t(n) = 0.4n + 5.4$ , so solve  $0.4n + 5.4 \geq 24.99$ .  $n = 48.975$ . In 49 weeks he will have \$25. If he must also cover tax, he will need another three or four weeks.
20. This is a function, it represents an arithmetic sequence, the graph is discrete but the points are linear. The domain is the positive integers: 1, 2, 3, .... The range is the sequence itself: 14, 10, 6, 2,  $-2, \dots$ . There are no asymptotes.

21. 3, 8, 13, 18, 23
22. 2, -1, -4, -7, -10
23.  $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}$
24. 5, 8, 11, 14, 17
25. 5, 8, 11, 14, 17
26. 5, 2, -1, -4, -7
27. -3, 3, 9, 15, 21
28.  $\frac{1}{3}, \frac{5}{6}, 1\frac{1}{3}, 1\frac{5}{6}, 2\frac{1}{3}$
29. 148
30. 0
31. 48
32. -62
33.  $t(n) = 4n$  ;  $t(1) = 4, t(n+1) = t(n) + 4$
34.  $t(n) = 7n - 9$  ;  $t(1) = -2, t(n+1) = t(n) + 7$
35.  $t(n) = 39 - 12n$ ;  $t(1) = 27, t(n+1) = t(n) - 12$
36.  $a_n = \frac{1}{3}n + 2\frac{2}{3}$  ;  $a_1 = 3, a_{n+1} = a_n + \frac{1}{3}$
37. graph (21): points (1, 3), (2, 8), (3, 13), (4, 18), (5, 23) slope = 5  
graph (22): points (1, 2), (2, -1), (3, -4), (4, -7), (5, -10) slope = -3
38. The slope of the line containing the points is the same as the common difference of the sequence.
39. 10, 20, 40, 80, 160
40. -9, -27, -81, -243, -729
41. 40, 20, 10, 5,  $\frac{5}{2}$
42. 6, -3,  $\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}$
43. 5, 15, 45, 135, 405
44. 100, 50, 25,  $\frac{25}{2}, \frac{25}{4}$
45. -3, 6, -12, 24, -48
46.  $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$
47. 464
48. 16
49. 6
50.  $\frac{27}{2}$
51.  $t(n) = \frac{2}{5} \cdot 5^n$  ;  $t(1) = 2, t(n+1) = t(n) \cdot 5$
52.  $t(n) = 64 \cdot \left(\frac{1}{4}\right)^n$  ;  $t(1) = 16, t(n+1) = t(n) \cdot \frac{1}{4}$
53.  $t(n) = \frac{5}{3} \cdot 3^n$  ;  $t(1) = 5, t(n+1) = t(n) \cdot 3$
54.  $t(n) = \frac{-3}{2} \cdot (-2)^n$  ;  $t(1) = 3, t(n+1) = t(n) \cdot (-2)$
55. Graph (39): Points on curve through (1, 10), (2, 20), (3, 40), (4, 80), and (5, 160).  
Graph (52): Points on curve through (1, 16), (2, 4), (3, 1), (4,  $\frac{1}{4}$ ), and (5,  $\frac{1}{16}$ ).
56. Arithmetic sequences are linear and geometric sequences are curved (exponential).